

Course material

Course:

Digital Signal Processing II

Video:

2.3.1.c Power Spectral Density.c

Concepts (extracted from automatically generated subtitles):

Time n. Signal generation. Sum of the square value of each sample. Coin toss. Power density. Spectral properties of this signal. Sample value. Dft of this portion of the process. Random processes. Interval of length n. Spectral representation. Minus $j\omega k$. Expected value. Spectral density of a white noise process. Signal generator capital m times.



[to video sequence search](#)
(within Digital Signal Processing II.)



[to video](#)

Center for Digital Education. More educational support material here:

<https://exoset.epfl.ch/booc/>

spectral representation of random processes

In this lecture, will try to derive a spectral representation for random processes.

notes

summary

0m 0s



For each new sample, toss a fair coin:

$$x[n] = \begin{cases} +1 & \text{if the outcome of the } n\text{-th toss is head} \\ -1 & \text{if the outcome of the } n\text{-th toss is tail} \end{cases}$$

- ▶ each sample is independent from all others
- ▶ each sample value has a 50% probability

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So let's start with the coin toss and example that we have used before. Imagine that we have a machine that each time n flips a coin and outputs 1, if the outcome is head and minus 1, if the outcome is tail. We assume that each sample is independent and each sample value clearly if the coin is fair, has 50% probability of being plus 1 minus 1.

notes

summary

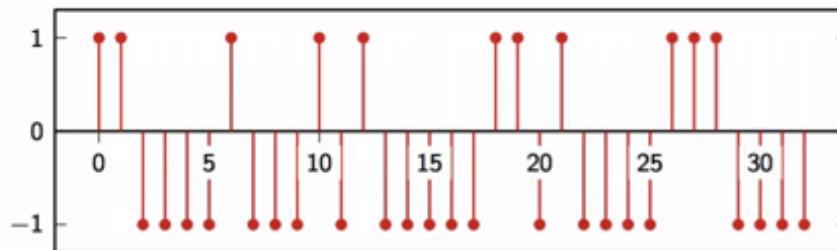
0m 7s



A simple discrete-time random signal generator



every time we turn on the generator we obtain a different *realization* of the signal



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So we can run this machine say for instance here, we run it for 32 time indices. And every time we run the machine, we obtain a different realization of the signal each realization is an alternation of plus 1 and minus 1 in a random order. And each time we run the machine, we get 32 new samples that are different every time.

notes

summary

0m 30s



A simple discrete-time random signal generator



- ▶ every time we turn on the generator we obtain a different *realization* of the signal
- ▶ we know the “mechanism” behind each instance
- ▶ but how can we analyze a random signal? What about its frequency content?

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So we know the mechanism behind the signal generation, but how could we represent the spectral properties of this signal? What is the frequency domain representation of this random sequence?

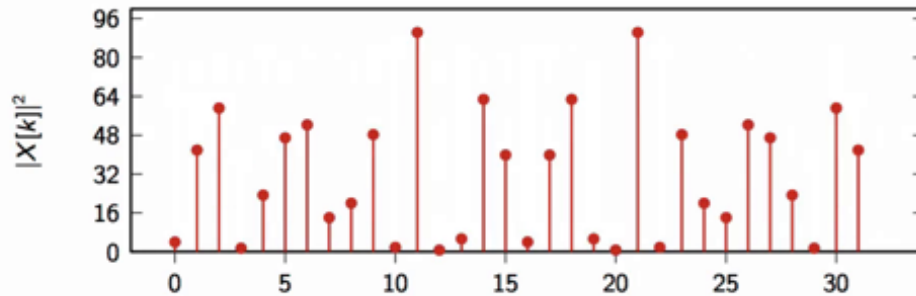
notes

summary

0m 50s



let's try with the DFT of a finite set of random samples



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The first thing we could do is try and simply to take the DFT of this portion of the process that we are considering. So we have 32 samples, we take the DFT and here we're looking at the magnitude of the DFT. We don't really see much. Of course, the DFT is symmetric because the signal is real valued but there is no apparent pattern. And if we run the signal generation machine once again,

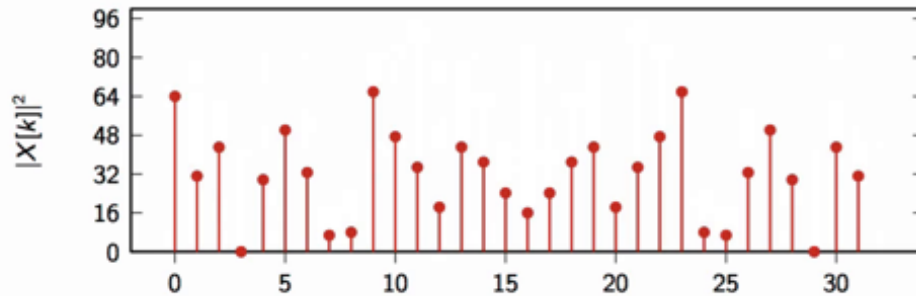
notes

summary

1m 5s



let's try with the DFT of a finite set of random samples



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and we perform the DFT, we get a completely different shape once again. So it doesn't seem like taking the DFT of a portion of a realization of a random process can highlight any particular spectral properties of the process itself. So perhaps we might think if we increase the length of the portion that we analyzed and we take in more points, maybe the DFT will be clearer.

notes

summary

1m 33s



every time it's different; maybe with more data?



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Maybe the DFT will highlight some of the properties.

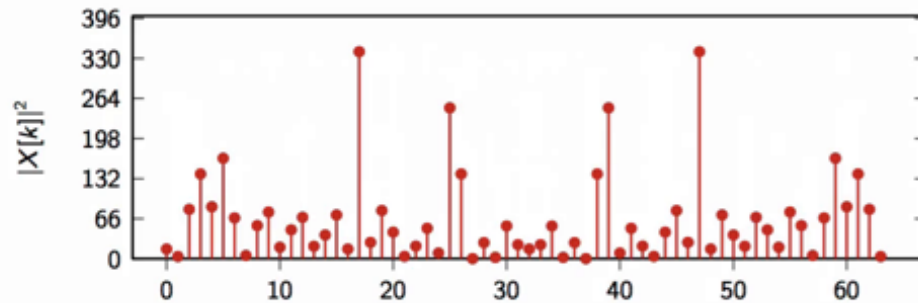
notes

summary

2m 0s



DFT of an increasing number of samples



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So let's take a longer portion of the random process. So here we've been doing our plots with 32 samples. Let's try with 64 samples and nothing really seems to appear in the plot.

notes

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summary

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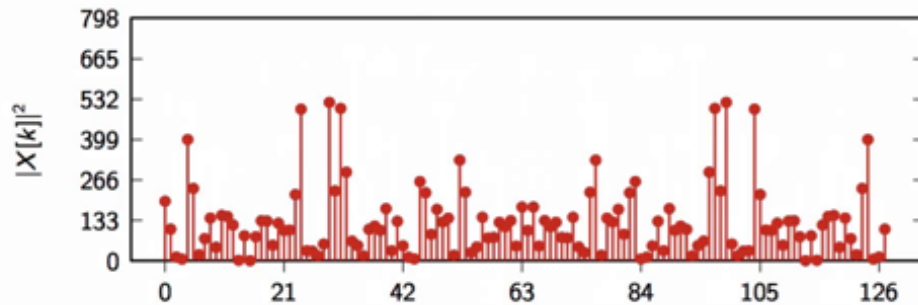
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2m 1s



DFT of an increasing number of samples



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We can increase the data length even more. Here, we have a 128 points but again, there is no pattern in the DFT. So clearly, taking the DFT of a single realization is not the way to go.

notes

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summary

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2m 13s



- ▶ DFTs of realizations show no clear pattern... we need a new strategy
- ▶ when faced with random data an intuitive response is to take “averages” (i.e. expectation)
- ▶ for the coin-toss signal:

$$E[x[n]] = -1 \cdot P[\text{n-th toss is tail}] + 1 \cdot P[\text{n-th toss is head}] = 0$$

- ▶ so the average value for each sample is zero...

Okay, so when we have something that is random and we want to mitigate the effects of randomness the standard strategy is to try and take averages. Now the problem is that the coin toss signal we just defined is 0 mean, namely the expected value for each sample is going to be 0. So if the average value of the signal is 0,

notes

summary

2m 27s



- ▶ ... as a consequence, averaging the DFT will not work
- ▶ $E[X[k]] = 0$
- ▶ however the signal "moves", so its energy or power must be nonzero

the average value of the DFT will be 0 as well because of linearity.

notes

summary

2m 48s



- ▶ ... as a consequence, averaging the DFT will not work
- ▶ $E[X[k]] = 0$
- ▶ however the signal "moves", so its energy or power must be nonzero

But the signal does move it moves from minus 1 to 1 randomly, but it does move. So the energy or the power if you want that is contained in a realization is nonzero. So we should be able to see this power in the frequency domain.

notes

summary

2m 49s



- ▶ the coin-toss signal has infinite energy:

$$E_x = \lim_{N \rightarrow \infty} \underbrace{\sum_{n=-N}^N |x[n]|^2}_{= 2N+1} = \lim_{N \rightarrow \infty} (2N+1) = \infty$$

- ▶ however it has finite power over any interval:

$$P_x = \lim_{N \rightarrow \infty} \underbrace{\frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2}_{= 1} = 1$$

Let's try and dig into this intuition a bit more. In general, if we have a full realization of the coin toss signal, the energy associated to the realization is going to be infinite. If we take the sum of the square value of each sample, well, that is going to be equal to the number of samples. And therefore, as the support over which we compute the energy grows to infinity, the energy itself will go to infinity. However, the power of each realization is going to be constant and equal to 1 over any interval over which we compute the power. So the coin toss signal is a power signal.

notes

summary

3m 8s



- ▶ pick an interval length N
- ▶ pick a number of iterations M
- ▶ run the signal generator M times and obtain M N -point realizations
- ▶ compute the DFT of each realization
- ▶ average their square magnitude divided by N

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notes

3m 45s



let's try to average the DFT's square magnitude, normalized:

- ▶ pick an interval length N
- ▶ pick a number of iterations M
- ▶ run the signal generator M times and obtain M N -point realizations
- ▶ compute the DFT of each realization
- ▶ average their square magnitude divided by N

We pick an interval of length n , we pick a number of iterations m that corresponds to the number of realizations that we will use to average the DFT over.

notes

summary

3m 58s



let's try to average the DFT's square magnitude, normalized:

- ▶ pick an interval length N
- ▶ pick a number of iterations M
- ▶ run the signal generator M times and obtain M N -point realizations
- ▶ compute the DFT of each realization
- ▶ average their square magnitude divided by N

We run the signal generator capital M times and we obtain capital M and point realizations. We compute the DFT of each realization.

notes

summary

4m 8s



let's try to average the DFT's square magnitude, normalized:

- ▶ pick an interval length N
- ▶ pick a number of iterations M
- ▶ run the signal generator M times and obtain M N -point realizations
- ▶ compute the DFT of each realization
- ▶ average their square magnitude divided by N

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And then we square the DFT and we compute the average of all this capital M DFTs. So let's do this numerically and see how this looks.

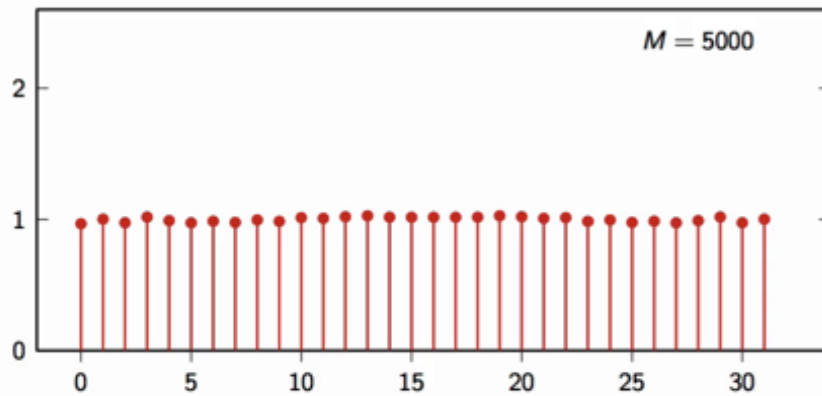
notes

summary

4m 17s



Averaged DFT square magnitude



$N = 32$

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So if we start with one single realization, we get this random pattern that we have seen before. But as we increase the number of realizations, we see that the average value of the squared DFT coefficient seems to converge to 1 for all values of the index. So in this example, we have taken capital N equal to 32.

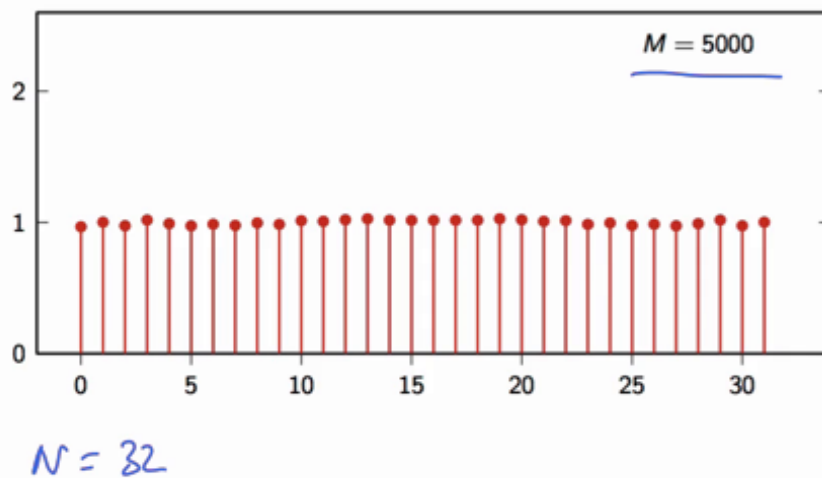
notes

summary

4m 25s



Averaged DFT square magnitude



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So 32 points of each realization, and now we have 5,000 different realizations that were using to compute this average.

notes

summary

4m 48s



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- 44



$$P[k] = E [|X_N[k]|^2 / N]$$

↑ —

- ▶ it looks very much as if $P[k] = 1$
- ▶ if $|X_N[k]|^2$ tends to the *energy* distribution in frequency...
- ▶ ... $|X_N[k]|^2 / N$ tends to the *power* distribution (aka *density*) in frequency
- ▶ the frequency-domain representation for stochastic processes is the power spectral density

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magnitude of the DFT coefficient number K normalized by the length of the DFT. From the previous plot, it looks that the sequence P of K converges to 1 if we take enough repetitions of the experiment.

notes

summary

5m 2s



$$P[k] = E [|X_N[k]|^2 / N]$$

- ▶ it looks very much as if $P[k] = 1$
- ▶ if $|X_N[k]|^2$ tends to the *energy* distribution in frequency...
- ▶ ... $|X_N[k]|^2 / N$ tends to the *power* distribution (aka *density*) in frequency
- ▶ the frequency-domain representation for stochastic processes is the power spectral density

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Now, if the square magnitude of a DFT coefficient usually represents the energy in frequency of a signal. The square magnitude normalized by the length of the analysis interval represents the power distribution or the power density in frequency for the signal. And so the frequency domain representation for stochastic processes seems to be the power spectral density. We will analyze this in more detail in just a second.

notes

summary

5m 16s



- ▶ $P[k] = 1$ means that the power is equally distributed over all frequencies
- ▶ i.e., we cannot predict if the signal moves "slowly" or "super-fast"
- ▶ this is because each sample is independent of each other: we could have a realization of all ones or a realization in which the sign changes every other sample or anything in between

45

The intuition, however, behind the fact that P_k is equal to 1 for the coin toss signal is that it seems that the power of this stochastic process is equally distributed over all frequencies.

notes

summary

5m 43s



- ▶ $P[k] = 1$ means that the power is equally distributed over all frequencies
- ▶ i.e., we cannot predict if the signal moves "slowly" or "super-fast"
- ▶ this is because each sample is independent of each other: we could have a realization of all ones or a realization in which the sign changes every other sample or anything in between

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And this makes sense intuitively because the signal is random so we could get a sequence of all ones or a sequence of all ones with correspond to power allocated exclusively to the slowest frequency. Or we could get a sequence of perfectly alternating plus 1 and minus 1 which would correspond to power allocated to the highest frequency and every combination in between. So the power spectral density of the signal seems to reflect the fact that we cannot predict the shape of the signal. But we do know that the signal alternates between plus 1 and minus 1.

notes

summary

5m 58s



- ▶ let's filter the random process with a 2-point Moving Average filter
- ▶ $y[n] = (x[n] + x[n - 1])/2$
- ▶ what is the power spectral density?

A variation on the theme is considering a filtered random process.

notes

summary

6m 35s



- ▶ let's filter the random process with a 2-point Moving Average filter
- ▶ $y[n] = (x[n] + x[n - 1])/2$
- ▶ what is the power spectral density?

So let's take a very, very simple filter, a two-point moving average and we use this filter to filter the coin toss process that we just defined.

notes

summary

6m 37s



- ▶ let's filter the random process with a 2-point Moving Average filter
- ▶ $y[n] = (x[n] + x[n - 1])/2$
- ▶ what is the power spectral density?

So the output of the filter will be just the sum of two consecutive samples divided by 2.
What is the power spectral density of a filtered random process?

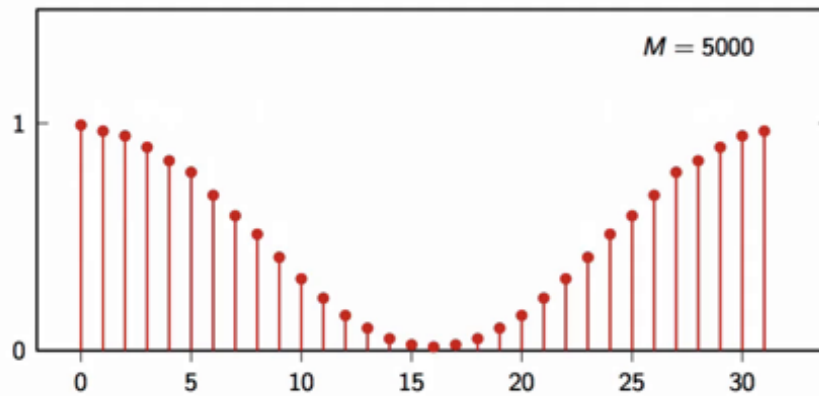
notes

summary

6m 49s



Averaged DFT magnitude of filtered process



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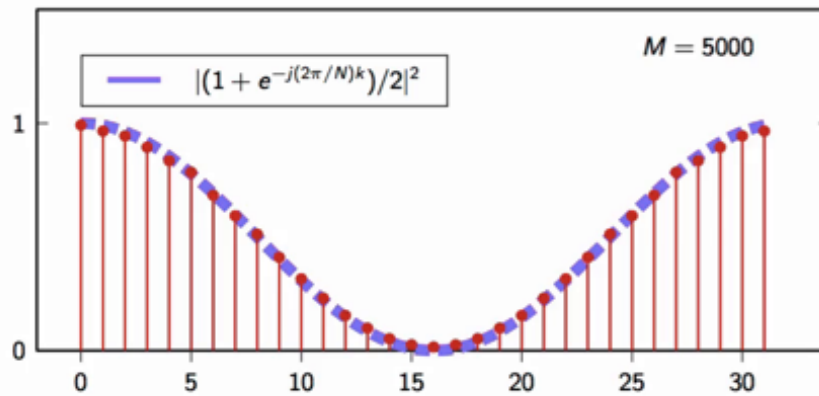
So we repeat numerically the same experiment with it before. We generate capital M realizations of the coin toss process. We take say for instance, a 32-point segment of this realizations. And we filter these with a two-point moving average. Then we take the DFT, we square the coefficients. We normalize it by the length of the DFT and we average this result over different realizations. So if you take a single realization, we have a pattern doesn't tell us much but as we increase the number of realizations. We see that once again, the plot of the average squared DFT coefficients seem to converge to a clear pattern.

notes

summary

6m 59s





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And with 5,000 realizations like before, we have a clear curve here that is exactly corresponding to the square magnitude of the frequency response of the filter.

notes

summary

7m 44s



- ▶ it looks like $P_y[k] = P_x[k] |H[k]|^2$, where $H[k] = \text{DFT} \{h[n]\}$
- ▶ can we generalize these results beyond a finite set of samples?

So it appears that for filtered process, the power spectral density of the output is equal to the power spectral density of the input multiplied by the square magnitude of the frequency response of the filter. Well all this is very nice but of course we derive these results for a very specific random process and a very specific filter. And so now our goal is to show that these results hold in general. Before we do so, we have to go back a bit to one of the earliest lectures in this

notes

summary

7m 53s



► energy signals: $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

► power signals: $\lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2 < \infty$

DSP course and recall the difference between energy and power signals. Energy signals are signals that have a finite square norm. You take the sum of the square absolute value of each sample over the entire support of the signal and the result is finite. Power signals on the other hand, have infinite energy, but they have a finite rate of energy. So if you take the energy over an interval and you normalize that by the length of the interval you always have a finite number.

notes

summary

8m 21s



- ▶ finite support, $\text{sinc}(n)$, $\alpha^n u[n]$ for $|\alpha| < 1$, ...
- ▶ DTFT is well defined
- ▶ DTFT square magnitude is *energy* distribution in frequency

We have encountered many examples of energy signals so all finite support signals are finite energy. Of course, the sink is a finite energy signal and exponential sequences with a ratio less than 1 in magnitude are also finite energy signals. For this signals the, DFT is well-defined we have no problem computing it. And the square magnitude of the DTFT represents

notes

summary

8m 49s



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notes

9m 17s



Consider a truncated DTFT

$$X_N(e^{j\omega}) = \sum_{n=-N}^N x[n]e^{-j\omega n}$$

define the power spectral density of a signal as:

$$P(e^{j\omega}) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} |X_N(e^{j\omega})|^2$$

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namely the power spectral density. To define the power spectral density, consider that truncated DTFT sum so you have your DTFT sum and you range the index from some minus n integer to n.

notes

summary

9m 56s



Consider a truncated DTFT

$$X_N(e^{j\omega}) = \sum_{n=-N}^N x[n]e^{-j\omega n}$$

define the power spectral density of a signal as:

$$P(e^{j\omega}) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} |X_N(e^{j\omega})|^2$$

52

And you define the power spectral density as the square magnitude of this truncated DTFT normalized by the length of the support over which you compute this truncated DTFT. And then you take the limit as n goes to infinity.

notes

summary

10m 11s



Examples:

- ▶ $x[n] = a, P_x(e^{j\omega}) = a^2 \tilde{\delta}(\omega)$
- ▶ $x[n] = au[n], P_x(e^{j\omega}) = (a^2/2) \tilde{\delta}(\omega)$
- ▶ $x[n] = ae^{j\sigma n}, P_x(e^{j\omega}) = a^2 \tilde{\delta}(\omega - \sigma)$

53

The result is a representation of power in frequency for a power signal. So for example, the constant signal x of n equal to a has a power spectral density,

notes

summary

10m 30s



Examples:

- ▶ $x[n] = a, P_x(e^{j\omega}) = a^2 \tilde{\delta}(\omega)$
- ▶ $x[n] = au[n], P_x(e^{j\omega}) = (a^2/2) \tilde{\delta}(\omega)$
- ▶ $x[n] = ae^{j\sigma n}, P_x(e^{j\omega}) = a^2 \tilde{\delta}(\omega - \sigma)$

53

which is equal to a squared times a Direct Delta centered in 0. The step sequence x a scholar a has a power spectral

notes

summary

10m 40s



Examples:

- ▶ $x[n] = a, P_x(e^{j\omega}) = a^2 \tilde{\delta}(\omega)$
- ▶ $x[n] = au[n], P_x(e^{j\omega}) = (a^2/2) \tilde{\delta}(\omega)$
- ▶ $x[n] = ae^{j\sigma n}, P_x(e^{j\omega}) = a^2 \tilde{\delta}(\omega - \sigma)$

density which is equal to a squared over 2 x a direct Delta n 0. A complex exponential on the other hand will have a power spectral density which is the square of the magnitude of the complex exponential times a Dirac Delta

notes

summary

10m 52s



For a random process

$$P_x(e^{j\omega}) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} E[|X_N(e^{j\omega})|^2]$$

54

centered at the frequency of the complex exponential. Now, a wide sense stationary process with finite variance is a power signal. So its frequency domain representation will be a power spectral density.

notes

summary

11m 3s



For a random process

$$P_x(e^{j\omega}) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \underbrace{\mathbb{E} [|X_N(e^{j\omega})|^2]}$$

54

We just need to modify the definition we just introduced before by replacing the square magnitude of the truncated

notes

summary

11m 13s



For a random process

$$P_x(e^{j\omega}) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \underbrace{\mathbb{E} [|X_N(e^{j\omega})|^2]}$$

54

DTFT with the expected value of say squared magnitude.

notes

summary

11m 25s



For a random process

$$P_x(e^{j\omega}) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \underbrace{\mathbb{E} [|X_N(e^{j\omega})|^2]}$$

54

This takes into account the randomness of the signal and averages out across realizations the randomness of the DTFT.

notes

summary

11m 27s



$$\begin{aligned} E \left[\left| \sum_{n=-N}^N x[n] e^{-j\omega n} \right|^2 \right] &= E \left[\sum_{n=-N}^N x[n] e^{j\omega n} \sum_{m=-N}^N x[m] e^{-j\omega m} \right] \\ &= \sum_{n=-N}^N \sum_{m=-N}^N E[x[n]x[m]] e^{-j\omega(m-n)} \end{aligned}$$



Okay, so let's compute this term. We can write the expected value of the magnitude of the truncated DTFT as the expected value of the truncated DTFT multiplied by the conjugate of the truncated DTFT. We assume the signal is real so we only need to conjugate the complex exponential here. Now, we collect the sums and

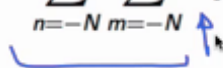
notes

summary

11m 38s



$$\begin{aligned} E \left[\left| \sum_{n=-N}^N x[n] e^{-j\omega n} \right|^2 \right] &= E \left[\sum_{n=-N}^N x[n] e^{j\omega n} \sum_{m=-N}^N x[m] e^{-j\omega m} \right] \\ &= \sum_{n=-N}^N \sum_{m=-N}^N E[x[n]x[m]] e^{-j\omega(m-n)} \end{aligned}$$



we move the expectation operator inside the sums because of linearity.

notes

summary

11m 59s



$$\begin{aligned}
 E \left[\left| \sum_{n=-N}^N x[n] e^{-j\omega n} \right|^2 \right] &= E \left[\sum_{n=-N}^N x[n] e^{j\omega n} \sum_{m=-N}^N x[m] e^{-j\omega m} \right] \\
 &= \sum_{n=-N}^N \sum_{m=-N}^N E[x[n]x[m]] e^{-j\omega(m-n)} \\
 &= \sum_{n=-N}^N \sum_{m=-N}^N r_x[m-n] e^{-j\omega(m-n)}
 \end{aligned}$$

WSS

And now we use the wide sense stationarity hypothesis for the process and this expectation here, expectation of the product between X of N and X of M where n and M are the indices of the two sums, becomes simply the autocorrelation of the process computed at time lag M minus n .

notes

summary

12m 4s



$$\begin{aligned}
 E \left[\left| \sum_{n=-N}^N x[n] e^{-j\omega n} \right|^2 \right] &= E \left[\sum_{n=-N}^N x[n] e^{j\omega n} \sum_{m=-N}^N x[m] e^{-j\omega m} \right] \\
 &= \sum_{n=-N}^N \sum_{m=-N}^N E[x[n]x[m]] e^{-j\omega(m-n)} \\
 &= \sum_{n=-N}^N \sum_{m=-N}^N \boxed{r_x[m-n]} e^{-j\omega(m-n)}
 \end{aligned}$$

WSS

And we have of course always a complex exponential here. Now here, we have a double sum and we're going to use a very clever trick to transform this double sum into a single sum. Let's see how to do that.

notes

summary

12m 24s



$$S = \sum_{m=-N}^N \sum_{n=-N}^N f(m-n)$$

$$-2N \leq (m-n) \leq 2N$$

$$S = \sum_{k=-2N}^{2N} c(k)f(k)$$

So suppose we have a double sum of this form. We have an outer index that goes from minus big N to big N and

notes

summary

12m 34s



$$S = \sum_{m=-N}^N \sum_{n=-N}^N \underbrace{f(m-n)}$$

$$-2N \leq (m-n) \leq 2N$$

$$S = \sum_{k=-2N}^{2N} c(k)f(k)$$

an inner index that also goes from minus big N and to big N. And the function inside double sum only depends on the difference between the summation indices.

notes

summary

12m 44s



$$S = \sum_{m=-N}^N \sum_{n=-N}^N f(m-n)$$

↴

$$\underbrace{-2N}_{\text{blue}} \leq (m-n) \leq \underbrace{2N}_{\text{blue}}$$

$$S = \sum_{k=-2N}^{2N} c(k)f(k)$$

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So the range of this argument here $m-n$ if you consider the range of the summation indices goes from minus two times capital N plus 2 times capital N .

notes

summary

12m 53s



$$S = \sum_{m=-N}^N \sum_{n=-N}^N f(m-n)$$

$$-2N \leq (m-n) \leq 2N$$

$$S = \sum_{k=-2N}^{2N} c(k)f(k)$$

So we can rewrite this double sum as a single sum where the index now ranges from minus to capital M to capital N. And the function inside the sum is computed over the index and each term F of K is multiplied by a weighing factor here C of K.

notes

summary

13m 5s



$$S = \sum_{m=-N}^N \sum_{n=-N}^N f(m-n)$$

$$-2N \leq (m-n) \leq 2N$$

$$S = \sum_{k=-2N}^{2N} c(k)f(k)$$

Which will take into account the number of times that the term M minus N is equal to K in the original summation.

notes

summary

13m 21s



$c(k)$: number of ways we can pick n, m in $[-N, N]$ so that $(m - n) = k$



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C of K, therefore is the number of ways that we can pick and N and M in the interval minus big N to big N so that M minus n is equal to K.

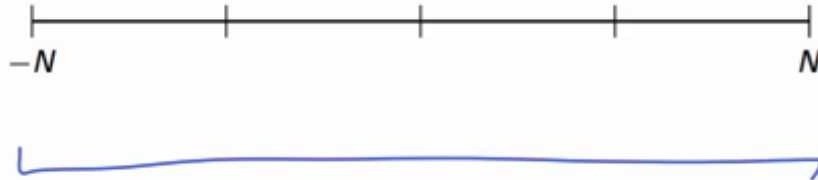
notes

summary

13m 28s



geometrically: $c(k)$ = number of ways we can fit a segment of length k over $[-N, N]$:



So if you think about it a bit, this is geometrically equivalent to the number of ways that we can fit a segment of length K over an interval that goes from minus n to big N .

notes

summary

13m 37s



A number line diagram illustrating the relationship between n and m . The number line has tick marks at $-N$, 0 , and N . A red bracket is drawn from $-N$ to 0 , labeled n . Another red bracket is drawn from 0 to m , labeled m . The segment from $-N$ to m is the total length $n + m$.

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notes

13m 49s



$c(k)$: number of ways we can pick n, m in $[-N, N]$ so that $(m - n) = k$

$$c(k) = 2N + 1 - |k|$$

58

After some reflection, we can come up with a formula for the number of ways that we can do so. And C of K turns out to be 2 times capital N plus 1 minus the absolute value of K .

notes

summary

14m 4s



$$E \left[\left| \sum_{n=-N}^N x[n] e^{-j\omega n} \right|^2 \right] = \sum_{k=-2N}^{2N} (2N+1-|k|) r_x[k] e^{-j\omega k}$$

So we use this to turn that expected squared DFT value into a single sum. So here we have what we were looking for and this is the current state of the equation. We have a sum that goes from minus 2 times capital N to 2 times capital N of this wave factor here to n plus 1 minus absolute value of K times the autocorrelation of the input process computing K times e to the minus J Omega K. This starts look very much like a DTFT.

notes

summary

14m 23s



$$\begin{aligned}
 P_x(e^{j\omega}) &= \lim_{N \rightarrow \infty} \sum_{k=-2N}^{2N} \left(\frac{2N+1-|k|}{2N+1} \right) (r_x[k]e^{-j\omega k}) \\
 &= \lim_{N \rightarrow \infty} \sum_{k=-2N}^{2N} w_N[k] r_x[k] e^{-j\omega k}
 \end{aligned}$$

But let's move on. Remember, we're looking for the power spectral density, which is the expected value of the squared DTFT normalized by the length of the interval over which we computed the partial DTFT. And that length is 2 capital N plus 1, so this is now what we're having.

notes

summary

14m 50s



$$\begin{aligned}
 P_x(e^{j\omega}) &= \lim_{N \rightarrow \infty} \sum_{k=-2N}^{2N} \left(\frac{2N+1-|k|}{2N+1} \right) (r_x[k]e^{-j\omega k}) \\
 &= \lim_{N \rightarrow \infty} \sum_{k=-2N}^{2N} w_N[k] r_x[k] e^{-j\omega k}
 \end{aligned}$$

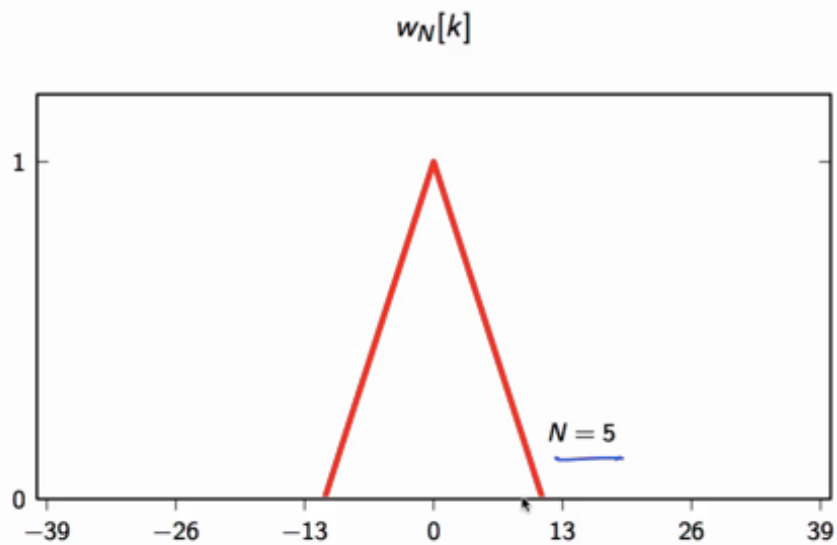
This term here, let's call it $w_N[k]$ is a weighing factor that multiplies the value of the autocorrelation in k and the complex exponential in $\minus j \Omega \omega k$. And we want to take this limit for big N goes to infinity.

notes

summary

15m 10s





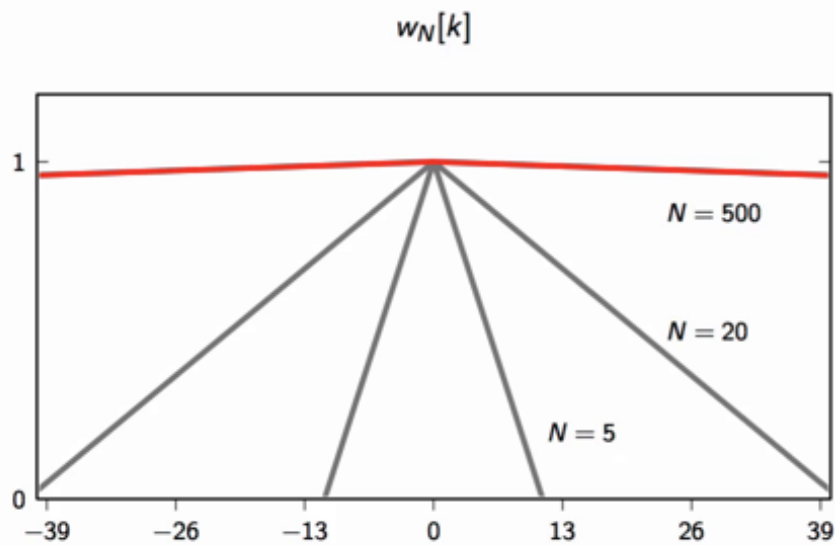
Let's examine how this guy behaves as n grows to infinity. If we plot W_N of K for all possible values of K . Well here, we take a short windows from minus 40 to 40, the function looks

notes

summary

15m 33s





like a triangle and as n increases, this triangle becomes wider and wider.

notes

summary

15m 46s



$$\begin{aligned}
 P_x(e^{j\omega}) &= \lim_{N \rightarrow \infty} \sum_{k=-2N}^{2N} w_N[k] r_x[k] e^{-j\omega k} \\
 &= \sum_{k=-\infty}^{\infty} r_x[k] e^{-j\omega k} \\
 &= \text{DTFT}\{r_x[k]\}
 \end{aligned}$$

As n goes to infinity w_n of K will tend to the constant 1 for all values of K . So with this in mind, we compute the limit that we were looking for, okay.

notes

summary

15m 49s



$$\begin{aligned}
 P_x(e^{j\omega}) &= \lim_{N \rightarrow \infty} \sum_{k=-N}^{2N} \cancel{w_N[k]} \cancel{r_x[k]} e^{-j\omega k} \\
 &= \sum_{k=-\infty}^{\infty} r_x[k] e^{-j\omega k} \\
 &= \text{DTFT}\{r_x[k]\}
 \end{aligned}$$

So the limit for n that goes to Infinity of this some, the weighing factor, the autocorrelation, the complex exponential. As n goes to Infinity, this goes to 1 and we're left with the sum for

notes

summary

16m 1s



$$\begin{aligned}
 P_x(e^{j\omega}) &= \lim_{N \rightarrow \infty} \sum_{k=-2N}^{2N} \cancel{w_N[k]} \cancel{r_x[k]} e^{-j\omega k} \\
 &= \sum_{k=-\infty}^{\infty} r_x[k] e^{-j\omega k} \\
 &= \text{DTFT}\{r_x[k]\}
 \end{aligned}$$

K that goes from minus infinity to plus infinity of the autocorrelation

notes

summary

16m 12s



$$\begin{aligned}
 P_x(e^{j\omega}) &= \lim_{N \rightarrow \infty} \sum_{k=-2N}^{2N} \cancel{w_N[k]} \cancel{r_x[k]} e^{-j\omega k} \\
 &= \sum_{k=-\infty}^{\infty} r_x[k] e^{-j\omega k} \\
 &= \text{DTFT}\{r_x[k]\}
 \end{aligned}$$

of the input process in K times e to the minus J Omega K. Which is the DTFT of the autocorrelation of the input process. So this is a remarkable result the power spectral density

notes

summary

16m 13s



White noise:

- ▶ $m = 0$
- ▶ $r[k] = \sigma^2 \delta[k]$

$$P(e^{j\omega}) = \sigma^2$$

of a random process is the DTFT of its autocorrelation. As an example, let's look at the power spectral density of a white noise process. Remember, a white noise process of defined by a mean equal to 0 and auto correlation that is nonzero only for K equal to 0 in which case it's equal to the variance of the process. Well, if you take the DTFT of the autocorrelation for the white noise

notes

summary

16m 34s



White noise:

- ▶ $m = 0$
- ▶ $r[k] = \sigma^2 \delta[k]$

$$P(e^{j\omega}) = \sigma^2$$

process, you have the power spectral density is the constant Sigma square. Where Sigma square is the variance of the process. Now, the coin toss process, you remember, was white because every coin toss is independent and it's autocorrelation is Delta of K. And therefore, we have confirmed that the power spectral density for the coin toss process is indeed a constant.

notes

summary

16m 58s



- 67

notes

17m 20s

