

Course material

Course:

Digital Signal Processing II

Video:

2.3.1.d Filtering Random Processes

Concepts (extracted from automatically generated subtitles):

N minus k.. Expectation of the sample y. Linear time-invariant filter. Take-home lesson. Convolution sum. N minus m. Impulse response h. Wide sense. Expected value. Random process. Values of the impulse response. Impulse response. K times. Linear combination of input samples. Input indices n.



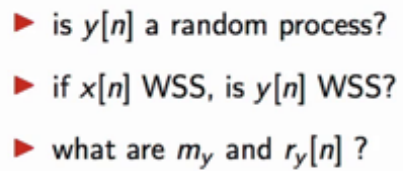
[to video sequence search](#)
(within Digital Signal Processing II.)



[to video](#)

Center for Digital Education. More educational support material here:

<https://exoset.epfl.ch/booc/>

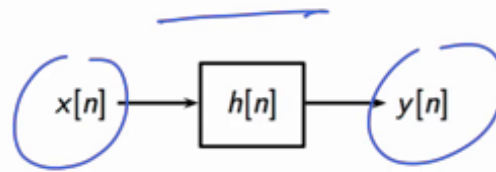


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notes

0m 0s





- ▶ is $y[n]$ a random process?
- ▶ if $x[n]$ WSS, is $y[n]$ WSS?
- ▶ what are m_y and $r_y[n]$?

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Y of N and the questions that we're going to ask and find an answer to are is Y of N a random process? If so, assuming that the input is wide sense stationary, is the output wide-sense stationary? What are therefore the mean and the auto-correlation of the output sequence? So the first question is rather easy. Y of N is obtained as a linear combination of input samples weighed by the values of the impulse response. So if the input is random the output will be random as well.

notes

summary

0m 13s



$$\begin{aligned}
 m_{y[n]} &= E[y[n]] = E\left[\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right] \\
 &= \sum_{k=-\infty}^{\infty} h[k]E[x[n-k]] \\
 &= \sum_{k=-\infty}^{\infty} h[k]m_x \quad (x[n] \text{ is WSS}) \\
 &= m_x \sum_{k=-\infty}^{\infty} h[k] \\
 &= m_x H(e^{j0})
 \end{aligned}$$

Let's now try and compute the mean of the output random process.

notes

summary

0m 45s



$$\begin{aligned}
 \underbrace{m_{y[n]}} &= \underbrace{E[y[n]]} = E \left[\sum_{k=-\infty}^{\infty} h[k] \underbrace{x[n-k]} \right] \\
 &= \sum_{k=-\infty}^{\infty} h[k] E[x[n-k]] \\
 &= \sum_{k=-\infty}^{\infty} h[k] m_x \quad (x[n] \text{ is WSS}) \\
 &= m_x \sum_{k=-\infty}^{\infty} h[k] \\
 &= m_x H(e^{j0})
 \end{aligned}$$

We write M of Y of N because we cannot assume that the mean is time-invariant. So we take the expectation of the sample Y of N and we replace Y of N by the convolution sum. So the convolution between

notes

summary

0m 49s



$$\begin{aligned}
 m_{y[n]} &= E[y[n]] = E \left[\sum_{k=-\infty}^{\infty} h[k]x[n-k] \right] \\
 &= \sum_{k=-\infty}^{\infty} h[k] E[x[n-k]] \\
 &= \sum_{k=-\infty}^{\infty} h[k] m_x \quad (x[n] \text{ is WSS}) \\
 &= m_x \sum_{k=-\infty}^{\infty} h[k] \\
 &= m_x H(e^{j0})
 \end{aligned}$$

the impulse response and the input process. The expectation is a linear operator so we can pass it inside the sum and we have H of K and the expected value of X of N minus K . But the input is wide sense stationary, so this expected value does not depend on the time index, is the mean of the input process which is time-invariant. So this sum becomes the sum from K that goes from minus infinity to plus infinity of H of K times the mean of the input.

notes

summary

1m 6s



$$\begin{aligned}
 m_{y[n]} &= E[y[n]] = E\left[\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right] \\
 &= \sum_{k=-\infty}^{\infty} h[k]E[x[n-k]] \\
 &= \sum_{k=-\infty}^{\infty} h[k]m_x \quad (x[n] \text{ is WSS}) \\
 &= m_x \sum_{k=-\infty}^{\infty} h[k] \\
 &= m_x H(e^{j0}) \quad \omega=0
 \end{aligned}$$

Which we course we can take out of the sum and we're left with the mean of the input then multiplies the sum of all the values of the impulse response or alternatively the value of the frequency response for omega equal to zero. So the mean of the output is also time-invariant.

notes

summary

1m 37s



$$\begin{aligned}
 E[y[n]y[m]] &= E\left[\sum_{k=-\infty}^{\infty} h[k]x[n-k] \sum_{i=-\infty}^{\infty} h[i]x[m-i]\right] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]E[x[n-k]x[m-i]] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]r_x[(n-m)-(k+i)]
 \end{aligned}$$

output depends only on lag $(n - m) \rightarrow y[n]$ is WSS

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Let's see about the other correlations. So we take the expected value of Y of N and Y of

notes

summary

1m 57s



$$\begin{aligned}
 E[y[n]y[m]] &= E \left[\sum_{k=-\infty}^{\infty} h[k]x[n-k] \sum_{i=-\infty}^{\infty} h[i]x[m-i] \right] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]E[x[n-k]x[m-i]] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]r_x[(n-m)-(k+i)]
 \end{aligned}$$

output depends only on lag $(n - m) \rightarrow y[n]$ is WSS

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M for two generic values of N and M. We replace Y of N and Y of M with the explicit convolution sum and then we rearrange

notes

summary

2m 1s



$$\begin{aligned}
 E[y[n]y[m]] &= E \left[\sum_{k=-\infty}^{\infty} h[k]x[n-k] \sum_{i=-\infty}^{\infty} h[i]x[m-i] \right] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]E[x[n-k]x[m-i]] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]r_x[(n-m) - (k+i)]
 \end{aligned}$$

output depends only on lag $(n - m) \rightarrow y[n]$ is WSS

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the terms and we pass the expectation operator inside. So we have a double sum here, where we have H of K and H of I and K of I being the indices of the sum. Here because of the wide sense stationarity of the input, we have the auto-correlation computed in N minus M minus K plus I. Now, in this sum,

notes

summary

2m 13s



$$\begin{aligned}
 E[y[n]y[m]] &= E \left[\sum_{k=-\infty}^{\infty} h[k]x[n-k] \sum_{i=-\infty}^{\infty} h[i]x[m-i] \right] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]E[x[n-k]x[m-i]] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]r_x[(n-m) - (k+i)]
 \end{aligned}$$

output depends only on lag $(n-m) \rightarrow y[n]$ is WSS

K and I range from minus infinity to plus infinity.

notes

summary

2m 36s



$$\begin{aligned}
 E[y[n]y[m]] &= E \left[\sum_{k=-\infty}^{\infty} h[k]x[n-k] \sum_{i=-\infty}^{\infty} h[i]x[m-i] \right] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]E[x[n-k]x[m-i]] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]r_x[(n-m) - (k+i)]
 \end{aligned}$$

output depends only on lag $(n - m) \rightarrow y[n]$ is WSS

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So also this term will range from minus infinity to plus infinity. This expression only depends on

notes

summary

2m 37s



$$\begin{aligned}
 E[y[n]y[m]] &= E \left[\sum_{k=-\infty}^{\infty} h[k]x[n-k] \sum_{i=-\infty}^{\infty} h[i]x[m-i] \right] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]E[x[n-k]x[m-i]] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]r_x[(n-m) - (k+i)]
 \end{aligned}$$

output depends only on lag $(n - m) \rightarrow y[n]$ is WSS

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the input indices N and M via the difference and minus M,

notes

summary

2m 49s



$$\begin{aligned}
 E[y[n]y[m]] &= E \left[\sum_{k=-\infty}^{\infty} h[k]x[n-k] \sum_{i=-\infty}^{\infty} h[i]x[m-i] \right] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]E[x[n-k]x[m-i]] \\
 &= \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h[k]h[i]r_x((n-m) - (k+i))
 \end{aligned}$$

output depends only on lag $(n - m) \rightarrow y[n]$ is WSS

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namely on the time lag. So N and M appear in this expression only as a difference. So the output depends only on the time lag and therefore Y of N the output of a filtered wide sense stationary process is also wide sense stationary.

notes

summary

2m 50s



with a change of variable in the double sum:

$$r_y[n] = h[n] * h[-n] * r_x[n]$$

so that:

$$P_y(e^{j\omega}) = |H(e^{j\omega})|^2 P_x(e^{j\omega})$$

Deterministic filters can be used to shape the power distribution of WSS random processes

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With a change of variable in the double sum, we can easily show that

notes

summary

3m 9s



with a change of variable in the double sum:

$$r_y[n] = \underbrace{h[n]} * \underbrace{h[-n]} * \underbrace{r_x[n]}$$

so that:

$$P_y(e^{j\omega}) = |H(e^{j\omega})|^2 P_x(e^{j\omega})$$

Deterministic filters can be used to shape the power distribution of WSS random processes

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the auto-correlation of the output is equal to the convolution of the impulse response of the filter with the time-reversed impulse response

notes

summary

3m 13s



with a change of variable in the double sum:

$$r_y[n] = \underbrace{h[n]} * \underbrace{h[-n]} * \underbrace{r_x[n]} \quad]$$

so that:

$$\underbrace{P_y(e^{j\omega})} = |H(e^{j\omega})|^2 \underbrace{P_x(e^{j\omega})}$$

Deterministic filters can be used to shape the power distribution of WSS random processes

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and then convolved again with the input auto-correlation. Now if we take the DTFT of this expression, you obtain that the power spectral density of the output is equal to this power spectral density of the input times

notes

summary

3m 25s



with a change of variable in the double sum:

$$r_y[n] = \underbrace{h[n]} * \underbrace{h[-n]} * \underbrace{r_x[n]} \quad]$$

so that:

$$\underbrace{P_y(e^{j\omega})} = \underbrace{|H(e^{j\omega})|^2} \underbrace{P_x(e^{j\omega})}$$

Deterministic filters can be used to shape the power distribution of WSS random processes

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the squared magnitude response of the filter.

notes

summary

3m 36s



with a change of variable in the double sum:

$$r_y[n] = \underbrace{h[n]} * \underbrace{h[-n]} * \underbrace{r_x[n]} \quad]$$

so that:

$$\underbrace{P_y(e^{j\omega})} = \underbrace{|H(e^{j\omega})|^2}_{\downarrow} \underbrace{P_x(e^{j\omega})}$$

Deterministic filters can be used to shape the power distribution of WSS random processes

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What this means is that deterministic filters can be used to shape the power distribution of wide sense stationary random processes. What we lose in using filters with random processes,

notes

summary

3m 38s



with a change of variable in the double sum:

$$r_y[n] = \underbrace{h[n]} * \underbrace{h[-n]} * \underbrace{r_x[n]} \quad]$$

so that:

$$\underbrace{P_y(e^{j\omega})} = \underbrace{|H(e^{j\omega})|^2} \underbrace{P_x(e^{j\omega})}$$

Deterministic filters can be used to shape the power distribution of WSS random processes

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is the phase information. This makes sense once again, what we cannot predict about a random process is the shape in time but we can derive the distribution of power in frequency because the auto-correlation is a deterministic signal. So the magnitude of the frequency response of a filter will shape the power in frequency but we will have no control on the shape of the signal in time and that's why the phase information is lost.

notes

summary

3m 49s



key points:

- ▶ filters designed for deterministic signals still work (in magnitude) in the stochastic case
- ▶ we lose the concept of phase since we don't know the shape of a realization in advance

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The take-home lesson here is that the filters that we designed for deterministic signals will still work although only magnitude in the stochastic case. So that filter design techniques that we developed in the deterministic world are still applicable to random signals.

notes

summary

4m 17s

